

# Monotonicity of the collateralized debt obligations term structure model

Michał Barski

Faculty of Mathematics and Computer Science, University of Leipzig, Germany  
 Faculty of Mathematics, Cardinal Stefan Wyszyński University in Warsaw, Poland  
*Michal.Barski@math.uni-leipzig.de*

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## Abstract

The problem of existence of arbitrage free and monotone CDO term structure models is studied. Conditions for positivity and monotonicity of the corresponding Heath-Jarrow-Morton-Musiela equation for the  $x$ -forward rates with the use of the Milian type result are formulated. Two state spaces are taken into account - of square integrable functions and a Sobolev space. For the first the regularity results concerning pointwise monotonicity are proven. Arbitrage free and monotone models are characterized in terms of the volatility of the model and characteristics of the driving Lévy process.

**Key words:** CDO model, bond market, HJM condition, HJMM equation, monotonicity.

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**JEL Classification Numbers:** G10, G11

## 1 Introduction

A defaultable  $(T, x)$ -bond with maturity  $T > 0$  and credit rating  $x \in I \subseteq [0, 1]$ , a  $(T, x)$ -bond for short, is a financial contract which pays to its holder 1 Euro at time  $T$  providing that the writer of the bond hasn't bankrupted till time  $T$ . The set  $I$  above stands for all possible credit ratings. The bankruptcy is modeled with the use of a so called loss process  $\{L(t), t \geq 0\}$  which starts from zero, increases and takes values in the interval  $[0, 1]$ . The bond is worthless if the loss process exceeds its credit rating. Thus the payoff profile of the  $(T, x)$ -bond is of the form

$$\mathbf{1}_{\{L_T \leq x\}}.$$

The price  $P(t, T, x)$  of the  $(T, x)$ -bond is a stochastic process defined by

$$P(t, T, x) = \mathbf{1}_{\{L_t \leq x\}} e^{-\int_t^T f(t, u, x) du}, \quad t \in [0, T], \quad (1.1)$$

where  $f(\cdot, \cdot, x)$  stands for an  $x$ -forward rate. The value  $x = 1$  corresponds to the risk-free bond and  $f(t, T, 1)$  determines the *short rate* process via

$$f(t, t, 1), \quad t \geq 0.$$

The  $(T, x)$ -bond market is thus fully determined by the family of  $x$ -forward rates and the loss process  $L$ . This model is an extension of the classical non-defaultable bond market which can be identified with the case when  $I$  is a singleton, that is, when  $I = \{1\}$ .

The model of  $(T, x)$ -bonds above does not correspond to defaultable bonds which are directly traded on a real market. For instance, in this setting the bankruptcy of the  $(T, x_2)$ -bond automatically implies the bankruptcy of the  $(T, x_1)$ -bond if  $x_1 < x_2$ . In reality a bond with a higher credit rating may, however, default earlier than that with a lower one. The  $(T, x)$ -bonds were introduced in [3] as basic instruments related to the pool of defaultable assets called Collateralized Debt Obligations (CDO), which are actually widely traded on the market. In the CDO market model the loss process  $L(t)$  describes the part of the pool which has defaulted up to time  $t > 0$  and  $F(L_T)$ , where  $F$  is some function, specifies the CDO payoff at time  $T > 0$ . In particular,  $(T, x)$ -bonds may be identified with the digital-type CDO payoffs with the function  $F$  of the form

$$F(z) = F_x(z) := \mathbf{1}_{[0, x]}(z), \quad x \in I, z \in [0, 1].$$

Then the price of that payoff  $p_t(F_x(L_T))$  at time  $t \leq T$  equals  $P(t, T, x)$ . Moreover, as was shown in [3], each regular CDO claim can be replicated, and thus also priced, with a portfolio consisting of a certain combination of  $(T, x)$ -bonds. Thus it follows that the model of  $(T, x)$ -bonds determines the structure of the CDO payoffs. The induced family of prices

$$P(t, T, x), \quad T \geq 0, x \in I,$$

will be called a *CDO term structure model* or briefly a *CDO model*.

On real markets the price of a claim which pays more is always higher. This implies

$$P(t, T, x_1) = p_t(F_{x_1}(L_T)) \leq p_t(F_{x_2}(L_T)) = P(t, T, x_2), \quad t \in [0, T], \quad x_1 < x_2, \quad x_1, x_2 \in I, \quad (1.2)$$

which means that the prices of  $(T, x)$ -bonds are increasing in  $x$ . Similarly, if the claim is paid earlier, then it has a higher value and hence

$$P(t, T_1, x) = p_t(F_x(L_{T_1})) \geq p_t(F_x(L_{T_2})) = P(t, T_2, x), \quad t \in [0, T_1], \quad T_1 < T_2, \quad x \in I, \quad (1.3)$$

which means that the  $(T, x)$ -bond prices are decreasing in  $T$ . The CDO term structure model is called *monotone* if both conditions (1.2), (1.3) are satisfied. Surprisingly, monotonicity of the  $(T, x)$ -bond prices is not always preserved in mathematical models even if they satisfy severe no-arbitrage conditions, see [3] p.60. The aim of the paper is to specify the CDO term structure models which are arbitrage-free and monotone. That problem is also studied in [14] but in different model settings and with the use of different methods than presented in this paper.

We consider a finite family of  $x$ -forward rates in the Musiela parametrization

$$r(t, z, x_i) := f(t, t + z, x_i), \quad t, z \geq 0, x_i \in I,$$

with  $I = \{0 \leq x_1 < x_2 < \dots < x_n = 1\}$  and study the existence of arbitrage free and monotone CDO models specified by the family of prices

$$P(t, T, x_i) = \mathbf{1}_{\{L_t \leq x_i\}} e^{-\int_0^{T-t} r(t, u, x_i) du}, \quad t, T \geq 0, x_i \in I.$$

The forward rate dynamics is given by a stochastic partial differential equation (SPDE) of the form

$$dr(t) = (Ar(t) + F(t, r(t)))dt + G(t, r(t-))dZ(t), \quad t \geq 0, \quad (1.4)$$

where  $A$  is a differential operator:  $Ah(z, x_i) = \frac{\partial}{\partial z}h(z, x_i)$  and  $Z$  is a one dimensional Lévy process. The drift  $F$  is determined by the volatility  $G$  and the Laplace exponent of the process  $Z$ , via the generalized version of the Heath-Jarrow-Morton condition. Solutions of the equation (1.4), called a Heath-Jarrow-Morton-Musiela equation, are assumed to take values in the Hilbert spaces  $\mathbb{L}_n^{2,\gamma}$  or  $\mathbb{H}_n^{1,\gamma}$ , which means that  $r(t, \cdot, x_i)$ , for each  $x_i \in I$ , is a square integrable function, resp. belongs to the Sobolev space of functions with square integrable first derivative. From the results in [15], see also [3], one can deduce that existence of an arbitrage free CDO model is equivalent to the solvability of (1.4) and *pointwise monotonicity at zero* of the solution, i.e.

$$r(t, 0, x_i) \geq r(t, 0, x_{i+1}), \quad i = 1, 2, \dots, n-1, \quad \text{for almost all } t \geq 0. \quad (1.5)$$

Our approach is based on examining positivity and monotonicity in  $x_i \in I$  of the solution to (1.4). Generalizing the result of Milian, see [8], which originally deals with the Wiener process driven SPDEs, we deduce conditions on the volatility  $G$  and jumps of the Lévy process which are equivalent to positivity and monotonicity of the  $\mathbb{L}_n^{2,\gamma}$ -valued forward rate solving (1.4). These are conditions (P1), (P2), (M1), (M2), see Section 4 for a precise formulation, which show that  $G$  must satisfy certain growth and Lipschitz-type conditions with constants dependent on possible jumps of the process  $Z$ . Monotonicity of  $r$  in  $\mathbb{L}_n^{2,\gamma}$  does not imply (1.5), because  $r$  does not have to be pointwise well defined. However, we show that under square integrability condition for  $Z$  the solution of (1.4) actually satisfies (1.5) and thus automatically generates an arbitrage free CDO model. Its monotonicity follows from the positivity and monotonicity of the  $x$ -forward rates. These results are formulated as Theorem 4.1 and Proposition 4.3. The conditions providing arbitrage free and monotone CDO models generated by an  $\mathbb{H}_n^{1,\gamma}$ -valued solution of (1.4) are formulated in Theorem 4.2. In this case, as we show in Proposition 4.4, the regularity of elements of  $\mathbb{H}_n^{1,\gamma}$  implies that positivity conditions (P1), (P2) are sufficient for the CDO model to be arbitrage free and monotone. We do not need (M1) nor (M2). The results mentioned above need the transformations  $F$  and  $G$  in (1.4) to have linear growth and satisfy linear growth conditions. The corresponding conditions in terms of the regularity of  $G$  in  $\mathbb{L}_n^{2,\gamma}$ , resp.  $\mathbb{H}_n^{1,\gamma}$  and characteristics of the Lévy process are formulated in Proposition 4.5 and Proposition 4.6.

The paper is structured as follows. In Section 2 we present the preliminary results from [3] and [15] concerning absence of arbitrage in the CDO model. Here we follow the original papers and use standard parametrization for the  $x$ -forward rates. A precise formulation of the monotonicity problem involving the Heath-Jarrow-Morton-Musiela equation is presented in Section 3. Section 4 contains formulations of the main results that is Theorem 4.1 and Theorem 4.2 together with two auxiliary results - Proposition 4.3 and Proposition 4.4 concerning the problem of monotonicity and pointwise monotonicity. In Subsection 4.1 we present conditions for linear growth and local Lipschitz conditions which are needed in the main results. Further comments on positivity and monotonicity are presented in Subsection 4.2. Proofs are postponed to Section 5.

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## 2 No arbitrage conditions

To explain the model framework of the paper we compile preliminary results from [3] and [15]. They are concerned with the no-arbitrage conditions for the CDO market defined by the forward rates, in a standard parametrization, with the following dynamics

$$df(t, T, x_i) = a(t, T, x_i)dt + b(t, T, x_i)dZ(t), \quad t > 0, T > 0, x_i \in I, \quad (2.6)$$

where  $Z$  is a one dimensional Lévy process and  $I = \{x_1, x_2, \dots, x_n\}$  with  $0 \leq x_1 < x_2 < \dots < x_n = 1$ . Equation (2.6) can be treated as a system of stochastic equations parametrized by maturities  $T > 0$  and credit ratings  $x_i \in I$ . The model above was studied in [15] and also in [3] for the case when  $Z$  is a Wiener process. In the non-defaultable context, i.e. when  $I = \{1\}$  one obtains the classical bond market model setting introduced in [6].

The Lévy process  $Z$  admits the following Lévy-Itô decomposition

$$Z(t) = at + qW(t) + \int_0^t \int_{\{|y| \leq 1\}} y \hat{\pi}(ds, dy) + \int_0^t \int_{\{|y| > 1\}} y \pi(ds, dy), \quad t \geq 0, \quad (2.7)$$

where  $a \in \mathbb{R}$ ,  $q \geq 0$ ,  $W$  is a Wiener process and  $(\hat{\pi})$ ,  $\pi$  is a (compensated) Poisson jump measure of  $Z$ . Above  $\nu$  stands for the Lévy measure of  $Z$ , so it satisfies

$$\int_{\mathbb{R}} (|y|^2 \wedge 1) \nu(dy) < +\infty.$$

The characteristic triplet  $(a, q, \nu)$  determines the Lévy process in a unique way. The central role in the no-arbitrage conditions plays the Laplace transform  $J$  of  $Z$  which is defined by

$$E(e^{-zZ(t)}) = e^{tJ(z)}, \quad t \geq 0. \quad (2.8)$$

It is well known that the domain of  $J$  is of the form

$$B := \{z \in \mathbb{R} : \int_{\{|y| > 1\}} e^{-zy} \nu(dy) < +\infty\},$$

that is  $|J(z)| < +\infty$  if and only if  $z \in B$ , see [13], [10]. It follows that if  $B \neq \emptyset$  then some exponential moments of the Lévy process exist.

To formulate conditions which are equivalent to the absence of arbitrage on the CDO market, that is which ensure that the discounted bond prices

$$\hat{P}(t, T, x_i) := e^{-\int_0^t f(s, s, 1)ds} P(t, T, x_i) = e^{-\int_0^t f(s, s, 1)ds} \mathbf{1}_{\{L_t \leq x\}} e^{-\int_t^T f(t, u, x)du}, \quad T > 0, x_i \in I,$$

are local martingales, we need the following set of assumptions (A1)-(A3).

**(A1)** The *loss process*  $L$  is a càdlàg, non-decreasing, adapted, pure jump process of the form  $L_t = \sum_{s \leq t} \Delta L_s, t \geq 0$  with absolutely continuous compensator  $v(t, dx)dt$  satisfying  $\int_0^t v(s, I)ds < +\infty$ .

Under (A1) the process  $\mathbf{1}_{\{L_t \leq x_i\}}$  is càdlàg for each  $x_i \in I$  and has intensity of the form

$$\lambda(t, x_i) := v(t, (x_i - L_t, 1]),$$

that is the processes

$$\mathbf{1}_{\{L_t \leq x_i\}} - \int_0^t \mathbf{1}_{\{L_s \leq x_i\}} \lambda(s, x_i) ds,$$

is a martingale. Moreover,  $\lambda(t, x_i)$  is progressive and decreasing in  $x_i \in I$ .

**(A2)** For each  $(T, x_i)$  the coefficients  $a(t, T, x_i), b(t, T, x_i)$  are predictable and have bounded trajectories.

**(A3)** For each  $r > 0$  the function

$$u \rightarrow \int_{\{|y| > 1\}} e^{-uy} \nu(dy)$$

is bounded on the set  $\{u \in \mathbb{R} : |u| \leq r\} \cap B$ .

The following result comes from [15].

**Theorem 2.1** *Assume that (A1)-(A3) hold.*

a) *If  $\hat{P}(t, T, x_i), x_i \in I, T > 0$  are local martingales then*

$$\int_s^t b(s, u, x_i) du \in B \tag{2.9}$$

*for any  $0 \leq t \leq s$  on the set  $\{L_t \leq x_i\}$ ,  $dP \times dt$  a.s..*

b) *If (2.9) holds then  $\hat{P}(t, T, x_i), x_i \in I, T > 0$  are local martingales if and only if*

$$\int_t^s a(t, u, x_i) du = J \left( \int_t^s b(t, u, x_i) du \right), \tag{2.10}$$

$$f(t, t, x_i) = f(t, t, 1) + \lambda(t, x_i), \tag{2.11}$$

*for any  $0 \leq t \leq s$  on the set  $\{L_t \leq x_i\}$ ,  $dP \times dt$  a.s..*

If  $I$  is a singleton, i.e.  $I = \{1\}$  and  $W$  is a Wiener process then  $J(z) = \frac{1}{2}z^2$  and equation (2.10) reduces to the well known Heath-Jarrow-Morton condition from [6]. Differentiating (2.10) in  $s$  yields the explicit formula for the drift

$$a(t, T, x_i) = J' \left( \int_t^T b(t, u, x_i) du \right) b(t, T, x_i), \tag{2.12}$$

in terms of the volatility of the model. Equation (2.11) reflects the relation between the forward rate and the distribution of the loss process  $L$ . It follows that the loss process  $L$  may not be given a priori in an arbitrary way. In fact the loss process is uniquely determined by conditions

(2.10), (2.11). To see that we directly follow the arguments presented in [3]. Without loosing generality, we assume that the probability space has the following structure

$$\begin{aligned} (\mathbf{A4}) \quad \Omega &= \Omega_1 \times \Omega_2, \mathcal{F} = \mathcal{G} \otimes \mathcal{H}, \mathcal{F}_t = \mathcal{G}_t \otimes \mathcal{H}_t, P(d\omega) = P_1(d\omega_1)P_2(\omega_1, d\omega_2), \\ &\text{with } \omega = (\omega_1, \omega_2) \in \Omega, \end{aligned}$$

where  $(\Omega_1, \mathcal{G}, (\mathcal{G}_t), P_1)$  supports the Lévy process  $Z$  and  $\Omega_2$  is the canonical space of increasing,  $I$ -valued marked point functions endowed with filtration

$$\mathcal{H}_t := \sigma\{\omega_2(s) : s \leq t, \omega_2 \in \Omega_2\}, \mathcal{H} := \mathcal{H}_\infty.$$

Now one can fix paths of the loss process  $\omega_2(t) = L_t(\omega)$  and treat (2.6) with  $a$  satisfying (2.12) as an equation on  $(\Omega_1, \mathcal{G}, \mathcal{G}_t, P_1)$ . If this equation has a solution, then condition (2.11) can be written as

$$v(\omega, t, dx) = -f(\omega, t, t, \omega_2(t) + dx),$$

which means that the compensator of the loss process is determined by  $f$ . The problem of determining distribution of the process  $L$  is equivalent to finding the probability kernel  $P_2(\omega_1, d\omega_2)$  such that  $-f(\omega, t, t, \omega_2(t) + dx)$  actually forms a compensator. This holds if  $f(t, t, x_i)$  is decreasing in  $x_i$ . This leads to the following result which is a starting point for further analysis.

**Theorem 2.2** *Assume that (A1)-(A4) are satisfied. If (2.10) holds and (2.6) has a solution for each path of the loss process  $L_t$  such that the function*

$$x_i \longrightarrow f(t, t, x_i), \quad x_i \in I, \quad (2.13)$$

*is decreasing  $dP_1 \times dt$  a.s. and the process  $f(t, t, x_i)$  is progressive then the family  $\{f(t, T, x_i); \quad t, T \geq 0, x_i \in I\}$  forms an arbitrage free CDO model.*

### 3 Formulation of the problem

Here we reformulate the dynamics of the  $x$ -forward rate (2.6) by passing from the standard parametrization to the Musiela parametrization which was first used in [9]. For the running time  $t$  and maturity  $T$  one defines a new parameter  $z = T - t$  called *time to maturity*. Then the forward rates in Musiela parametrization are given by

$$r(t, z, x_i) := f(t, t + z, x_i), \quad t \geq 0, z \geq 0, x_i \in I.$$

and the induced bond prices by

$$P(t, T, x_i) = \mathbf{1}_{\{L_t \leq x_i\}} e^{-\int_0^{T-t} r(t, u, x_i) du}, \quad x_i \in I, T \geq 0. \quad (3.14)$$

Starting from (2.6) and using

$$G(t, z, x_i) := b(t, t + z, x_i), \quad F(t, z, x_i) := a(t, t + z, x_i).$$

we obtain

$$r(t)(z, x_i) = S_t(r_0)(z, x_i) + \int_0^t S_{t-s} F(s)(z, x_i) ds + \int_0^t S_{t-s} G(s)(z, x_i) dZ(s), \quad (3.15)$$

where  $S$  stands for the shift semigroup  $S_t(h)(z, x_i) := h(t + z, x_i)$ . This means that  $r$  is a weak solution of the equation

$$dr(t, z, x_i) = \left( Ar(t, z, x_i) + F(t, z, x_i) \right) dt + G(t, z, x_i) dZ(t), \quad t, T \geq 0, x_i \in I, \quad (3.16)$$

with a generator  $A$  of the semigroup  $S$  given by

$$Ar(t, z, x_i) := \frac{\partial r(t, z, x_i)}{\partial z}.$$

The volatility  $G$  in (3.16) is assumed to be a transformation of the form  $G(t, r(t-))$  with

$$G(t, \varphi)(z, x_i) = g(t, z, x_i, L_t, \varphi(z)), \quad t \geq 0, z \geq 0, \varphi = \varphi(z), \quad (3.17)$$

where  $L_t$  is a loss process and

$$g(\cdot, \cdot, x_i, \cdot, \cdot) =: g_i(\cdot, \cdot, \cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \times I \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad i = 1, 2, \dots, n, \quad (3.18)$$

is a sequence of functions. Since we are interested in arbitrage free models only, it follows from (2.12) that the drift coefficient  $F = F(t, r(t))$  in (3.16) is determined by

$$F(t, \varphi)(z, x_i) := J' \left( \int_0^z G(t, \varphi)(u, x_i) du \right) G(t, \varphi)(z, x_i), \quad t \geq 0, z \geq 0, \varphi = \varphi(z); x_i \in I. \quad (3.19)$$

The SPDE (3.16) with volatility  $G$  given by (3.17) and drift  $F$  of the form (3.19) will be called in the sequel a Heath-Jarrow-Morton-Musiela (HJMM) equation. It follows that the HJMM equation is specified by  $G$  and the function  $J'$  which in turn is determined by the characteristic triplet of the Lévy process. The HJMM equation in the non-defaultable context has been studied for instance in [1], [2], [4], [5], [7]. The state space for the solution of (3.16) is to be specified. To this end let us introduce two Hilbert spaces of measurable real valued functions defined on  $\mathbb{R}_+$ . The first consists of square integrable functions

$$L^{2,\gamma} := \left\{ h : \|h\|_{L^{2,\gamma}}^2 := \int_0^{+\infty} |h(u)|^2 e^{\gamma u} du < +\infty \right\},$$

and the second is the Sobolev space - a subspace of  $L^{2,\gamma}$  defined by

$$H^{1,\gamma} := \left\{ h : \|h\|_{H^{1,\gamma}}^2 := \int_0^{+\infty} (|h(u)|^2 + |h'(u)|^2) e^{\gamma u} du < +\infty \right\},$$

where  $\gamma > 0$ . The state spaces for the HJMM equation will be  $\mathbb{L}_n^{2,\gamma}$  and  $\mathbb{H}_n^{1,\gamma}$  consisting of functions  $h : \mathbb{R}_+ \times I \rightarrow \mathbb{R}$  such that  $h(\cdot, x_i) \in L^{2,\gamma}$ , resp.  $h(\cdot, x_i) \in H^{1,\gamma}$  for each  $x_i \in I$ . Endowed with the norms

$$\|h\|_{\mathbb{L}^{2,\gamma}}^2 := \sum_{i=1}^n \|h(\cdot, x_i)\|_{L^{2,\gamma}}^2, \quad \|h\|_{\mathbb{H}^{1,\gamma}}^2 := \sum_{i=1}^n \|h(\cdot, x_i)\|_{H^{1,\gamma}}^2.$$

they become Hilbert spaces.

In view of Theorem 2.2 the CDO model is arbitrage free if there exists a solution of the HJMM equation for each path of the loss process  $\{L_t, t \geq 0\}$  and such that it is *pointwise monotone at zero*, i.e.

$$r(t, 0, x_i) \geq r(t, 0, x_{i+1}), \quad i = 1, 2, \dots, n-1, \quad \text{for almost all } t \geq 0. \quad (3.20)$$

We additionally require that the  $(T, x_i)$ -bond prices, given by (3.14), are decreasing in  $T \geq 0$  and increasing in  $x_i \in I$ .

## 4 Formulation of the main results

Our conditions which characterize the arbitrage free and monotone CDO term structure models require the transformations  $G, F$ , given by (3.17) and (3.19), to be locally Lipschitz and to satisfy the linear growth condition (LGC) in  $\mathbb{H}$ , where  $\mathbb{H}$  stands for the state space, i.e. it is equal  $\mathbb{L}_n^{2,\gamma}$  or  $\mathbb{H}_n^{1,\gamma}$ . To be precise,  $F, G$  are *locally Lipschitz* (LC) if for any  $R > 0$  there exists  $C_R \geq 0$  such that

$$\|F(t, x) - F(t, y)\|_{\mathbb{H}} \leq C_R \|x - y\|_{\mathbb{H}}, \quad \|G(t, x) - G(t, y)\|_{\mathbb{H}} \leq C_R \|x - y\|_{\mathbb{H}} \quad (4.21)$$

for any  $x, y \in \mathbb{H}$  such that  $\|x\|_H, \|y\|_H \leq R$ , and satisfy *linear growth condition* if there exists  $C \geq 0$  such that

$$\|F(t, x)\|_{\mathbb{H}} \leq C \|x\|_{\mathbb{H}}, \quad \|G(t, x)\|_{\mathbb{H}} \leq C \|x\|_{\mathbb{H}} \quad (4.22)$$

for any  $x, y \in \mathbb{H}$ .

The first result is concerned with the space  $\mathbb{L}_n^{2,\gamma}$ . Recall, that  $\text{supp}\{\nu\}$  stands for the support of the Lévy measure.

**Theorem 4.1** *Let (A1) – (A4) be satisfied. Assume that  $F$  and  $G$  are locally Lipschitz transformations with linear growth in  $\mathbb{L}_n^{2,\gamma}$ . Then the following statements hold.*

- a) *For any path of the loss process there exists a unique weak solution to the HJMM equation in the space  $\mathbb{L}_n^{2,\gamma}$ .*
- b) *If for  $r = (r_1, r_2, \dots, r_n)$ ,  $r_1 \geq r_2 \geq \dots \geq r_n$ ,  $t, z \geq 0$ ,  $l \in I$ ,  $u \in \text{supp}\{\nu\}$ ,  $i = 1, 2, \dots, n-1$  hold*

$$(M1) \quad g_i(t, z, l, r) = g_{i+1}(t, z, l, r), \quad \text{if } r_i = r_{i+1},$$

$$(M2) \quad (g_{i+1}(t, z, l, r) - g_i(t, z, l, r))u \leq r_i - r_{i+1}.$$

and

$$\int_{\{|y| \geq 1\}} |y|^2 \nu(dy) < +\infty, \quad (4.23)$$

*then the solution of the HJMM equation is pointwise monotone at zero. Consequently the resulting CDO model is arbitrage free.*

- c) *If for  $r = (r_1, r_2, \dots, r_n)$ ,  $r \geq 0$ ,  $t, z \geq 0$ ,  $l \in I$ ,  $u \in \text{supp}\{\nu\}$ ,  $i = 1, 2, \dots, n$  hold*

$$(P1) \quad g_i(t, z, l, r) = 0, \quad \text{if } r_i = 0,$$

$$(P2) \quad r_i + g_i(t, z, l, r)u \geq 0.$$

*together with (M1), (M2) and (4.23), then the resulting CDO model is monotone.*

The second result is concerned with the  $\mathbb{H}_n^{1,\gamma}$ -valued forward rates.

**Theorem 4.2** *Let (A1) – (A4) be satisfied. Assume that  $F$  and  $G$  are locally Lipschitz transformations with linear growth in  $\mathbb{H}_n^{1,\gamma}$ . Then the following statements hold.*



- a) For any path of the loss process there exists a unique weak solution to the HJMM equation in the space  $\mathbb{H}_n^{1,\gamma}$ .
- b) If (P1) and (P2) hold then the solution of the HJMM equation is pointwise monotone at zero and hence the resulting CDO model is arbitrage free. Moreover, that model is monotone.

Both points (a) in Theorem 4.1 and Theorem 4.2 follow directly from the recent result on existence of solution of a general SPDE under locally Lipschitz condition and linear growth, see Theorem 4.1 in [2]. Section 4.1 is devoted to the direct specification of the volatility  $G$  of the HJMM equation and the characteristic triplet of the Lévy process for (4.21) and (4.22) to hold, see Proposition 4.5 and Proposition 4.6.

The pairs of conditions (P1), (P2) and (M1), (M2) correspond to positivity and monotonicity of the solution of the HJMM equation in  $\mathbb{L}_n^{2,\gamma}$ . They follow from a generalized version of the result of Milian, see [8], which was concerned with a general SPDE driven by a Wiener process. We show how to pass to a Lévy process in the case of the HJMM equation. To be more precise, we show in Theorem 5.3 in Section 5.2 that (M1), (M2) are equivalent to *monotonicity* of  $r$ , that is for each  $t \geq 0$

$$r(t, z, x_i) \geq r(t, z, x_{i+1}), \quad i = 1, 2, \dots, n-1, \quad (4.24)$$

holds for almost all  $z \geq 0$ , while (P1), (P2) to *positivity* of  $r$ , that is for each  $t \geq 0$

$$r(t, z, x_i) \geq 0, \quad x_i \in I, \quad (4.25)$$

holds for almost all  $z \geq 0$ . A delicate point here is the pointwise monotonicity of the solution at zero required for the CDO model to be arbitrage free. Actually (3.20) does not follow from (4.24). We call that problem *pointwise monotonicity* of the solution in  $\mathbb{L}_n^{2,\gamma}$  and solve by proving the following.

**Proposition 4.3** Assume that the transformations  $F, G : \mathbb{L}_n^{2,\gamma} \rightarrow \mathbb{L}_n^{2,\gamma}$  given by (3.19), (3.17) are locally Lipschitz and satisfy linear growth condition. Let  $Z$  satisfy

$$\int_{\{|y|>1\}} |y|^2 \nu(dy) < +\infty. \quad (4.26)$$

and the solution  $r(t), t \geq 0$  of (3.16) taking values in  $\mathbb{L}_n^{2,\gamma}$  be monotone. Then for each  $z \geq 0$ ,  $i = 1, 2, \dots, n-1$  holds

$$r(t, z, x_i) \geq r(t, z, x_{i+1}), \quad \text{for almost all } t \geq 0.$$

This result clearly implies monotonicity of  $r$  at zero and thus statement (b) in Theorem 4.1 follows. It is also clear that (4.24) and (4.25) imply monotonicity of the bond prices, so (c) in Theorem 4.1 holds. Notice that in Theorem 4.2 we do not require (M1) nor (M2). Of course, it follows from Theorem 4.1 (b) that if (M1) and (M2) hold then the  $\mathbb{H}_n^{1,\gamma}$ -valued solution is also monotone at zero because  $r(t, \cdot, x_i)$  is continuous. From continuity of elements in  $\mathbb{H}_n^{1,\gamma}$  follows, however, that the conditions (P1) and (P2) imply monotonicity of  $r$  at zero and monotonicity of the corresponding CDO model at once. More precisely, we prove the following

**Proposition 4.4** *Let  $r(t), t \geq 0$  be a positive solution of the HJMM equation in the space  $\mathbb{H}_n^{1,\gamma}$ . Then the bond prices  $P(t, T, x_i)$  are decreasing in  $T$ , increasing in  $x_i$  and  $r(t, 0, x_i)$  is decreasing in  $x_i$  on the set  $\{x_i : L_t \leq x_i\}$ .*

which implies the assertion (b) of Theorem 4.2.

In Section 4.2 we further comment on the conditions (P1), (P2), (M1), (M1) and give an example of a system of functions satisfying them. The detailed presentation dealing with the problem of monotonicity, positivity and pointwise monotonicity of the solution to the HJMM equation is contained in Section 5.2. There we start from the Milian theorem and, by using it to the HJMM equation, show validity of the conditions (P1), (P2), (M1), (M1) in Theorem 5.3. Afterwards we prove a sequence of auxiliary results which lead to the proofs of Proposition 4.3 and Proposition 4.4.

#### 4.1 Local Lipschitz conditions and linear growth

Here we formulate sufficient conditions for (4.21) and (4.22) to hold in  $\mathbb{L}_n^{2,\gamma}$  and  $\mathbb{H}_n^{1,\gamma}$ . For the space  $\mathbb{L}_n^{2,\gamma}$  we need the following regularity conditions for  $G$

##### Lipschitz condition

There exists a constant  $C > 0$  such that

$$(LC) \quad |g_i(t, z, l, r) - g_i(t, z, l, \bar{r})| \leq C \|r - \bar{r}\|, \quad t, z \geq 0, l \in I, r, \bar{r} \in \mathbb{R}^n.$$

##### boundedness condition

There exists  $\bar{g} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$(B1) \quad |g_i(t, z, l, r)| \leq \bar{g}(z), \quad t, z \geq 0, x_i, l \in I, r \in \mathbb{R}^n,$$

with  $K := \|\bar{g}\|_{\mathbb{L}_1^{2,\gamma}} < +\infty$ .

##### linear growth condition

There exists a constant  $C > 0$  such that

$$(LGC) \quad |g_i(t, z, l, r)| \leq C \|r\|, \quad t, z \geq 0, x_i, l \in I, r \in \mathbb{R}^n.$$

For the space  $\mathbb{L}_n^{2,\gamma}$  we have the following result.

**Proposition 4.5** *Assume that volatility  $G$  satisfies (LC) and*

*A) (B1) and one of the conditions*

- a)  $g_i \geq 0, i = 1, 2, \dots, n, \quad \text{supp}\{\nu\} \subseteq [-1, +\infty) \quad \text{and} \quad \int_1^{+\infty} |y|^2 \nu(dy) < +\infty,$
- b)

$$\int_{\{|y| \geq 1\}} y^2 e^{\frac{K}{\sqrt{\gamma}} y} \nu(dy) < +\infty,$$

where  $K$  is defined in (B1).

B) (LGC),  $g_i \geq 0, i = 1, 2, \dots, n$  and  $Z$  is such that

$$q = 0, \quad \text{supp}\{\nu\} \subseteq [0, +\infty), \quad \text{and} \quad \int_0^{+\infty} (|y| \vee |y|^2) \nu(dy) < +\infty. \quad (4.27)$$

Then  $F$  and  $G$  satisfy (4.21) and (4.22) in  $\mathbb{L}_n^{2,\gamma}$ .

For the results in the space  $\mathbb{H}_n^{1,\gamma}$  we need further regularity assumptions on  $G$ , that is, more restrictive *boundedness* conditions and conditions on the *derivatives* of  $\{g_i\}$ .

There exists a constant  $C > 0$  such that

$$(B2) \quad \hat{g} := \sup_z \bar{g}(z) < +\infty,$$

$$(B3) \quad |g(t, z, x_i, l, r)|^2 \leq C^2 \|r\|,$$

where  $\bar{g}(z)$  is defined in (B1).

For each  $i = 1, 2, \dots, n$ , the derivatives of  $g_i$  satisfy

$$(D1) \quad |g'_z(t, z, x_i, l, r) - g'_z(t, z, x_i, l, \bar{r})| + \|\nabla g_i(t, z, x_i, l, r) - \nabla g_i(t, z, x_i, l, \bar{r})\| \leq C \|r - \bar{r}\|,$$

$$(D2) \quad |g'_z(t, z, x_i, l, r)| \leq h(z), \quad t, z \geq 0, \quad x_i, l \in I, r \in \mathbb{R},$$

for some  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $h \in \mathbb{L}_1^{2,\gamma}$  and

$$(D3) \quad \sup_z |h(z)| < C,$$

and

$$(D4) \quad \sup_{t, z, l, r} \|\nabla g_i(t, z, x_i, l, r)\| < C,$$

where

$$\nabla g_i(t, z, l, r) := \begin{pmatrix} \frac{d}{dr_1} g_i(t, z, l, r) \\ \frac{d}{dr_2} g_i(t, z, l, r) \\ \dots \\ \frac{d}{dr_n} g_i(t, z, l, r) \end{pmatrix}, \quad i = 1, 2, \dots, n.$$

**Proposition 4.6** Assume that volatility  $G$  satisfies (LC), (D1) – (D4) and

A) (B1) – (B3) and one of the conditions

- a)  $g_i \geq 0, i = 1, 2, \dots, n, \quad \text{supp}\{\nu\} \subseteq [-1, +\infty) \quad \text{and} \quad \int_1^{+\infty} |y|^3 \nu(dy) < +\infty,$
- b)

$$\int_{\{|y| \geq 1\}} y^3 e^{\frac{K}{\sqrt{\gamma}} y} \nu(dy) < +\infty,$$

where  $K$  is defined in (B1).

B) (LGC), (B3) and  $g_i \geq 0, i = 1, 2, \dots, n$  together with

$$\sup_{t,z,l,r} |g_i(t, z, l, r)| < +\infty, \quad i = 1, 2, \dots, n.$$

Further, let  $Z$  be such that

$$q = 0, \quad \text{supp}\{\nu\} \subseteq [0, +\infty), \quad \text{and} \quad \int_0^{+\infty} (|y| \vee |y|^3) \nu(dy) < +\infty.$$

Then  $F$  and  $G$  satisfy (4.21) and (4.22) in  $\mathbb{H}_n^{1,\gamma}$ .

Let us comment the results above.

**Remark 4.7** The Lévy process satisfying condition (B) in Proposition 4.5 or (B) in Proposition 4.6 is a subordinator with drift, the Lévy measure of which additionally satisfies  $\int_{\{y>1\}} y^2 \nu(dy) < +\infty$ , resp.  $\int_{\{y>1\}} y^3 \nu(dy) < +\infty$ .

**Remark 4.8** If the jumps of the Lévy process  $Z$  are bounded then all the assumptions concerning the Lévy measure in Proposition 4.5 and Proposition 4.6 are satisfied.

The proofs of Proposition 4.5 and Proposition 4.6 are postponed to Section 5.1.

## 4.2 Further comments on positivity and monotonicity

Let us start with an observation concerning the case when the HJMM equation split into a separable system of equations.

**Remark 4.9** Assume that each function  $g_i$  in (3.16), as a function of  $r \in \mathbb{R}^n$ , depends on the  $i$ -th coordinate of  $r$  only, that is

$$g_i(t, z, l, r) = g_i(t, z, l, r_i), \quad t, z \geq 0, \quad l \in I, \quad i = 1, 2, \dots, n.$$

Then (M1) holds if and only if the system  $\{g_i\}_i$  reduces to one function, that is

$$g_i(t, z, l, r) = g_j(t, z, l, r), \quad t, z \geq 0, \quad l \in I,$$

for each  $i, j = 1, 2, \dots, n$ . This means that only a trivial system preserves monotonicity of forward rates.

Now we provide an auxiliary result dealing with conditions (P1), (P2), (M1), (M2). To abbreviate the notation set  $\mathbf{1}_i(r) := (r_1, r_2, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_n)$  for  $r \in \mathbb{R}^n$ .

**Proposition 4.10** A) Assume that  $g_i \geq 0, i = 1, 2, \dots, n$ .

a) If (P1) and (P2) hold, then

$$\text{supp}\{\nu\} \subseteq \left[ -\frac{1}{\sup_{t,z,l,r} g'_{r_i}(t, z, l, x_i, \mathbf{1}_i(r))}, +\infty \right), \quad (4.28)$$

for  $t, z \geq 0, l \in I, r = (r_1, r_2, \dots, r_n) \geq 0$  and  $i = 1, 2, \dots, n$ .

b) If (P1), (4.28) hold and

$$g_i(t, z, l, r) \leq g'_{r_i}(t, z, x_i, l, \mathbf{1}_i(r))r_i, \quad (4.29)$$

with  $g'_{r_i}(t, z, x_i, l, \mathbf{1}_i(r)) \geq 0$  for  $t, z \geq 0$ ,  $l \in I$ ,  $r = (r_1, r_2, \dots, r_n) \geq 0$   $i = 1, 2, \dots, n$ , then (P2) holds.

B) a) If (M1), (M2) hold, then for  $r = (r_1, r_2, \dots, r_n)$ ,  $r_1 \geq r_2 \geq \dots \geq r_n$ ,  $t, z \geq 0$ ,  $l \in I$ ,  $u \in \text{supp}\{\nu\}$ ,  $i = 1, 2, \dots, n-1$ , hold

$$\frac{d}{dr_i}[g_{i+1}(t, z, l, r) - g_i(t, z, l, r)]u \leq 1, \quad \text{for } r_i = r_{i+1}, \quad (4.30)$$

and

$$\frac{d}{dr_{i+1}}[g_{i+1}(t, z, l, r) - g_i(t, z, l, r)]u \leq 1, \quad \text{for } r_i = r_{i+1}, \quad (4.31)$$

b) Assume that (M1) and (4.30) hold. If, for  $r = (r_1, r_2, \dots, r_n)$ ,  $r_1 \geq r_2 \geq \dots \geq r_n$ ,  $t, z \geq 0$ ,  $l \in I$ ,  $u \in \text{supp}\{\nu\}$ ,  $i = 1, 2, \dots, n-1$ , one of the following conditions is satisfied

- (i)  $g_{i+1}(t, z, l, r) - g_i(t, z, l, r)$  is concave in  $r_i$  and  $\text{supp}\{\nu\} \subseteq (0, +\infty)$ ,
  - (ii)  $g_{i+1}(t, z, l, r) - g_i(t, z, l, r)$  is convex in  $r_i$  and  $\text{supp}\{\nu\} \subseteq (-\infty, 0)$ ,
  - (iii)  $g_{i+1}(t, z, l, r) - g_i(t, z, l, r)$  is concave in  $r_i$  and  $g_{i+1}(t, z, l, r) \geq g_i(t, z, l, r)$ ,
  - (iv)  $g_{i+1}(t, z, l, r) - g_i(t, z, l, r)$  is convex in  $r_i$  and  $g_{i+1}(t, z, l, r) \leq g_i(t, z, l, r)$ ,
- then (M2) holds.

With the use of Proposition 4.10 we can construct the following example.

**Example 4.11** Let us consider a system of functions of the multiplicative form

$$g_i(t, z, l, r) := f_1(t)f_2(z)f_3(l)h_1(r_1)h_2(r_2)\dots h_n(r_n)h(r_i); \quad i = 1, 2, \dots, n.$$

with smooth functions  $f_i, h_i, h$  and the following conditions

$$f_i, h_i, h \geq 0, \quad f_i \leq \bar{f}_i, h_i \leq \bar{h}_i, \quad \text{where } \bar{f}_i, \bar{h}_i \in \mathbb{R}_+ \quad i = 1, 2, \dots, n, \quad (4.32)$$

$$h(0) = 0, \quad h'(0) \geq 0, \quad h(r_i) \leq h'(0)r_i, \quad r_i \geq 0, \quad (4.33)$$

$$h_i \text{ is decreasing } i = 1, 2, \dots, n, \quad (4.34)$$

$$\text{supp}\{\nu\} \subseteq \left[-\frac{1}{a}, +\infty\right), \quad \text{with } a := \max_i \bar{f}_1 \bar{f}_2 \bar{f}_3 \bar{h}_1 \dots \bar{h}_{i-1} h_i(0) \bar{h}_{i+1} \dots \bar{h}_n h'(0), \quad (4.35)$$

$$0 \leq h' \leq \bar{h}', \quad \text{where } \bar{h}' \in \mathbb{R}_+, \quad \text{and } h, h_i \text{ are concave for } i = 1, 2, \dots, n, \quad (4.36)$$

$$\text{supp}\{\nu\} \subseteq \left[-\frac{1}{b}, +\infty\right), \quad \text{where } b := \bar{f}_1 \bar{f}_2 \bar{f}_3 \bar{h}_1 \dots \bar{h}_n \bar{h}'. \quad (4.37)$$

Then

a) (P1) and (P2) hold if (4.32)-(4.35) are satisfied.

b) (P1), (P2), (M1) and (M2) hold if (4.32)-(4.37) are satisfied.

The proof of Proposition 4.10 and calculations concerning Example 4.11 are postponed to Section 5.3.

## 5 Proofs

### 5.1 Local Lipschitz conditions and linear growth

Here we will prove Proposition 4.5 and Proposition 4.6. Let us start with the properties of the Laplace transform defined in (2.8). It is well known that  $J$  can be represented in the form

$$J(z) = -az + \frac{1}{2}qz^2 + \int_{\mathbb{R}} (e^{-zy} - 1 + zy\mathbf{1}_{(-1,1)}(y)) \nu(dy), \quad z \in B, \quad (5.38)$$

see [13], [10]. Moreover, the first and second derivative of  $J$  exist providing that corresponding exponential moment exist, see for instance [12]. In the sequel we will use the following result which can be proven directly.

**Lemma 5.1** *The functions  $J'$ ,  $J''$  are*

a) *continuous on  $[0, +\infty)$  if*

$$\text{supp}\{\nu\} \subseteq [-1, +\infty), \quad \text{and} \quad \int_1^{+\infty} |y|^2 \nu(dy) < +\infty.$$

b) *continuous on  $[-z_0, z_0]$  for some  $z_0 > 0$  if*

$$\int_{|y| \geq 1} |y|^2 e^{z_0|y|} \nu(dy) < +\infty.$$

c) *continuous and bounded on  $[0, +\infty)$  if  $Z$  does not contain the Wiener part, i.e.  $q = 0$ , and*

$$\text{supp}\{\nu\} \subseteq [0, +\infty), \quad \text{and} \quad \int_0^{+\infty} (|y| \vee |y|^2) \nu(dy) < +\infty.$$

First let us prove Proposition 4.5. For the sake of notational convenience all the estimations are presented in the equivalent coordinate form, that is for the transformations

$$\mathbb{L}_n^{2,\gamma} \ni \varphi(\cdot) \rightarrow G(t, \varphi)(\cdot, x_i) \in L^{2,\gamma}, \quad \mathbb{L}_n^{2,\gamma} \ni \varphi(\cdot) \rightarrow F(t, \varphi)(\cdot, x_i) \in L^{2,\gamma}, \quad i = 1, 2, \dots, n.$$

If (B1) holds, then for any  $\varphi : \mathbb{R} \rightarrow \mathbb{R}^n$  the following inequality holds

$$\begin{aligned} \int_0^z |g_i(t, v, L_t, \varphi(v))| dv &\leq \int_0^z \bar{g}(v) dv \leq \int_0^{+\infty} e^{-\frac{\gamma}{2}v} e^{\frac{\gamma}{2}v} \bar{g}(v) dv \\ &\leq \sqrt{\int_0^{+\infty} e^{-\gamma v} dv} \sqrt{\int_0^{+\infty} e^{\gamma v} \bar{g}^2(v) dv} \leq \frac{K}{\sqrt{\gamma}}. \end{aligned} \quad (5.39)$$

**Proof of Proposition 4.5:** (A) First we show linear growth. For any  $\varphi \in \mathbb{L}_n^{2,\gamma}$  we have

$$\|G(t, \varphi)(x_i)\|_{L^{2,\gamma}}^2 = \int_0^{+\infty} g_i^2(t, z, l, \varphi(z)) e^{\gamma z} dz \leq \int_0^{+\infty} \bar{g}^2(z) e^{\gamma z} dz = \|\bar{g}\|_{L^{2,\gamma}}^2.$$

(Aa) The assumption  $g \geq 0$  allows us to consider the function  $J'$  restricted to  $[0, +\infty)$ . It follows from condition (Aa) and Lemma 5.1 that  $J'$  is well defined and increasing on  $[0, +\infty)$ . Thus by (5.39) we have

$$\begin{aligned} \|F(t, \varphi)(x_i)\|_{L^{2,\gamma}}^2 &= \int_0^{+\infty} \left| J' \left( \int_0^z g_i(t, v, L_t, \varphi(v)) dv \right) g_i(t, z, L_t, \varphi(z)) \right|^2 e^{\gamma z} dz \\ &\leq \left| J' \left( \frac{K}{\sqrt{\gamma}} \right) \right|^2 \int_0^{+\infty} |\bar{g}(z)|^2 e^{\gamma z} dz = \left| J' \left( \frac{K}{\sqrt{\gamma}} \right) \right|^2 \|\bar{g}\|_{L^{2,\gamma}}^2. \end{aligned}$$

Let  $\varphi, \phi \in \mathbb{L}_n^{2,\gamma}$ . In view of (LC) we obtain

$$\begin{aligned} \|G(t, \varphi)(x_i) - G(t, \phi)(x_i)\|_{L^{2,\gamma}}^2 &= \int_0^{+\infty} \left| g_i(t, z, L_t, \varphi(z)) - g_i(t, z, L_t, \phi(z)) \right|^2 e^{\gamma z} dz \\ &\leq C^2 \int_0^{+\infty} \|\varphi(z) - \phi(z)\|^2 e^{\gamma z} dz = C^2 \|\varphi - \phi\|_{\mathbb{L}_n^{2,\gamma}}^2, \end{aligned} \quad (5.40)$$

and

$$\begin{aligned} &\|F(t, \varphi)(x_i) - F(t, \phi)(x_i)\|_{L^{2,\gamma}}^2 \\ &= \left\| J' \left( \int_0^z g_i(t, v, L_t, \varphi(v)) dv \right) g_i(t, z, L_t, \varphi(z)) - J' \left( \int_0^z g_i(t, v, L_t, \phi(v)) dv \right) g_i(t, z, L_t, \phi(z)) \right\|_{L^{2,\gamma}}^2 \\ &\leq 2I_1(x_i) + 2I_2(x_i), \end{aligned}$$

where

$$I_1(x_i) := \left\| J' \left( \int_0^z g_i(t, v, L_t, \varphi(v)) dv \right) \left| g_i(t, z, L_t, \varphi(z)) - g_i(t, z, L_t, \phi(z)) \right| \right\|_{L^{2,\gamma}}^2, \quad (5.41)$$

$$I_2(x_i) := \left\| g_i(t, z, L_t, \phi(z)) \left| J' \left( \int_0^z g_i(t, v, L_t, \varphi(v)) dv \right) - J' \left( \int_0^z g_i(t, v, L_t, \phi(v)) dv \right) \right| \right\|_{L^{2,\gamma}}^2. \quad (5.42)$$

It follows from Lemma 5.1 that under (Aa) the function  $J''$  is well defined and continuous on  $[0, +\infty)$ . Thus  $J'$  is Lipschitz on any interval  $[0, a]$ ,  $a > 0$ . Let  $C(J', \frac{K}{\sqrt{\gamma}})$  denote the Lipschitz constant of the function  $J'$  on the interval  $[0, \frac{K}{\sqrt{\gamma}}]$ . By (5.39) and (LC) we have

$$I_1(x_i) \leq \left| J' \left( \frac{K}{\sqrt{\gamma}} \right) \right|^2 C^2 \int_0^{+\infty} \|\varphi(z) - \phi(z)\|^2 e^{\gamma z} dz = \left| J' \left( \frac{K}{\sqrt{\gamma}} \right) \right|^2 C^2 \|\varphi - \phi\|_{\mathbb{L}_n^{2,\gamma}}^2,$$

$$\begin{aligned} I_2(x_i) &\leq \left\| g_i(t, z, L_t, \phi(z)) C \left( J', \frac{K}{\sqrt{\gamma}} \right) \int_0^{+\infty} |g_i(t, v, L_t, \varphi(v)) - g_i(t, v, L_t, \phi(v))| dv \right\|_{L^{2,\gamma}}^2 \\ &\leq \left\| g_i(t, z, L_t, \phi(z)) C \left( J', \frac{K}{\sqrt{\gamma}} \right) C \int_0^{+\infty} \|\varphi(v) - \phi(v)\| dv \right\|_{L^{2,\gamma}}^2 \\ &\leq C^2 \left( J', \frac{K}{\sqrt{\gamma}} \right) \frac{C^2}{\gamma} \|\bar{g}\|_{L^{2,\gamma}}^2 \|\varphi - \phi\|_{\mathbb{L}_n^{2,\gamma}}^2, \end{aligned}$$

and the assertion follows.

(Ab) If we drop the positivity assumption of  $\{g_i\}$ , then the proof above can be mimicked, but, in view of (5.39), we have to know that the function  $J'$  is well defined and Lipschitz on  $[-\frac{K}{\sqrt{\gamma}}, \frac{K}{\sqrt{\gamma}}]$  for each  $i = 1, 2, \dots, n$ . In view of Lemma 5.1 we need to assume (Ab).

(B) By (LGC) for any  $\varphi \in \mathbb{L}_n^{2,\gamma}$  holds

$$\|G(t, \varphi)(x_i)\|_{L^{2,\gamma}}^2 = \int_0^{+\infty} g_i^2(t, z, L_t, \varphi(z)) e^{\gamma z} dz \leq C^2 \int_0^{+\infty} \|\varphi(z)\|^2 e^{\gamma z} dz = C^2 \|\varphi\|_{\mathbb{L}_n^{2,\gamma}}^2.$$

The assumption  $g \geq 0$  allows us to consider the functions  $J', J''$  restricted to  $[0, +\infty)$ . It follows from Lemma 5.1 that if  $Z$  has no Wiener part and (4.27) holds, then  $J', J''$  are well defined on  $[0, +\infty)$  and bounded, i.e.

$$\sup_{z \geq 0} |J'(z)| \leq M, \quad \sup_{z \geq 0} |J''(z)| \leq M,$$

for some  $M > 0$ . Thus, in view of (LGC), we have

$$\begin{aligned} \|F(t, \varphi)(x_i)\|_{L^{2,\gamma}}^2 &= \int_0^{+\infty} \left| J' \left( \int_0^z g_i(t, v, L_t, \varphi(v)) dv \right) g_i(t, z, L_t, \varphi(z)) \right|^2 e^{\gamma z} dz \\ &\leq M^2 C^2 \int_0^{+\infty} \|\varphi(z)\|^2 e^{\gamma z} dz = M^2 C^2 \|\varphi\|_{\mathbb{L}_n^{2,\gamma}}^2. \end{aligned}$$

Let  $\varphi, \phi \in \mathbb{L}_n^{2,\gamma}$ . In view of (LC) we obtain

$$\begin{aligned} \|G(t, \varphi)(x_i) - G(t, \phi)(x_i)\|_{L^{2,\gamma}}^2 &= \int_0^{+\infty} \left| g_i(t, z, L_t, \varphi(z)) - g_i(t, z, L_t, \phi(z)) \right|^2 e^{\gamma z} dz \\ &\leq C^2 \int_0^{+\infty} \|\varphi(z) - \phi(z)\|^2 e^{\gamma z} dz = C^2 \|\varphi - \phi\|_{\mathbb{L}_n^{2,\gamma}}^2. \end{aligned} \quad (5.43)$$

To show that  $F$  satisfies local Lipschitz condition, first let us notice that, in view of (LGC), for any  $\|\varphi\|_{\mathbb{L}_n^{2,\gamma}} \leq R$  the following estimate holds

$$\begin{aligned} \int_0^z |g_i(t, v, L_t, \varphi(z))| dv &\leq C \int_0^z \|\varphi(v)\| dv \\ &\leq C \sqrt{\int_0^{+\infty} e^{-\gamma v} dv} \sqrt{\int_0^{+\infty} e^{\gamma v} \|\varphi(v)\|^2 dv} \leq \frac{CR}{\sqrt{\gamma}}, \quad z > 0. \end{aligned} \quad (5.44)$$

We have

$$\|F(t, \varphi)(x_i) - F(t, \phi)(x_i)\|_{L^{2,\gamma}}^2 \leq 2I_1(x_i) + 2I_2(x_i),$$

where  $I_1(x_i), I_2(x_i)$  are defined in (5.41), (5.42). Using (LC), (LGC) and (5.44) we obtain

$$I_1(x_i) \leq M^2 C^2 \int_0^{+\infty} \|\varphi(z) - \phi(z)\|^2 e^{\gamma z} dz = M^2 C^2 \|\varphi - \phi\|_{\mathbb{L}_n^{2,\gamma}}^2,$$



and

$$\begin{aligned}
I_2(x_i) &\leq \left\| g_i(t, z, L_t, \phi(z)) M \int_0^{+\infty} |g_i(t, v, L_t, \varphi(v)) - g_i(t, v, L_t, \phi(v))| dv \right\|_{L^{2,\gamma}}^2 \\
&\leq M^2 C^2 \left\{ \int_0^{+\infty} \|\varphi(v) - \phi(v)\| dv \right\}^2 C^2 \|\phi\|_{\mathbb{L}_n^{2,\gamma}}^2 \\
&\leq M^2 \frac{C^4}{\gamma} \|\varphi - \phi\|_{\mathbb{L}_n^{2,\gamma}}^2 \|\phi\|_{\mathbb{L}_n^{2,\gamma}}^2 \leq M^2 \frac{C^4}{\gamma} R^2 \|\varphi - \phi\|_{\mathbb{L}_n^{2,\gamma}}^2,
\end{aligned}$$

and thus local Lipschitz condition for  $F$  follows.  $\square$

Now we pass to the proof of Proposition 4.6. We examine the transformations

$$\mathbb{H}_n^{1,\gamma} \ni \varphi(\cdot) \rightarrow G(t, \varphi)(\cdot, x_i) \in H^{1,\gamma}, \quad \mathbb{H}_n^{1,\gamma} \ni \varphi(\cdot) \rightarrow F(t, \varphi)(\cdot, x_i) \in H^{1,\gamma}, \quad i = 1, 2, \dots, n,$$

and use the estimations from the proof of Proposition 4.5. Assume that  $F$  and  $G$  satisfy Lipschitz condition in  $\mathbb{L}_n^{2,\gamma}$ . Then it follows from the formula

$$\|h\|_{H^{1,\gamma}}^2 = \int_0^{+\infty} \left( h^2(z) + (h'(z))^2 \right) e^{\gamma z} dz = \|h\|_{L^{2,\gamma}}^2 + \|h'\|_{L^{2,\gamma}}^2,$$

that

$$\begin{aligned}
&\|G(t, \varphi)(\cdot, x_i) - G(t, \phi)(\cdot, x_i)\|_{H^{1,\gamma}}^2 \\
&= \|G(t, \varphi)(\cdot, x_i) - G(t, \phi)(\cdot, x_i)\|_{L^{2,\gamma}}^2 + \left\| \frac{d}{dz} G(t, \varphi)(\cdot, x_i) - \frac{d}{dz} G(t, \phi)(\cdot, x_i) \right\|_{L^{2,\gamma}}^2 \\
&\leq C \|\varphi - \phi\|_{\mathbb{L}_n^{2,\gamma}}^2 + \left\| \frac{d}{dz} G(t, \varphi)(\cdot, x_i) - \frac{d}{dz} G(t, \phi)(\cdot, x_i) \right\|_{L^{2,\gamma}}^2 \\
&\leq C \|\varphi - \phi\|_{\mathbb{H}_n^{1,\gamma}}^2 + \left\| \frac{d}{dz} G(t, \varphi)(\cdot, x_i) - \frac{d}{dz} G(t, \phi)(\cdot, x_i) \right\|_{L^{2,\gamma}}^2.
\end{aligned}$$

Thus to get Lipschitz conditions in  $\mathbb{H}_n^{1,\gamma}$  we will study transformations

$$\mathbb{H}_n^{1,\gamma} \ni \varphi \longrightarrow \frac{d}{dz} G(t, \varphi)(\cdot, x_i) \in L^{2,\gamma}, \quad \mathbb{H}_n^{1,\gamma} \ni \varphi \longrightarrow \frac{d}{dz} F(t, \varphi)(\cdot, x_i) \in L^{2,\gamma},$$

which, in view of (3.17) and (3.19), are given by

$$\frac{d}{dz} G(t, \varphi)(z, x_i) = g'_z(t, z, x_i, L_t, \varphi(z)) + \langle \nabla g_i(t, z, L_t, \varphi(z)), \varphi'(z) \rangle, \quad (5.45)$$

$$\begin{aligned}
\frac{d}{dz} F(t, \varphi)(z, x_i) &= J'' \left( \int_0^z g_i(t, u, L_t, \varphi(u)) du \right) g_i^2(t, z, L_t, \varphi(z)) \\
&\quad + J' \left( \int_0^z g_i(t, u, L_t, \varphi(u)) du \right) \left[ g'_z(t, z, x_i, L_t, \varphi(z)) + \langle \nabla g_i(t, z, L_t, \varphi(z)), \varphi'(z) \rangle \right].
\end{aligned} \quad (5.46)$$

Above  $\langle \cdot, \cdot \rangle$  stands for the scalar product in  $\mathbb{R}^n$ .

Let us start with an auxiliary inequality

$$\sup_{z \geq 0} \|\varphi(z)\| \leq \frac{2}{\sqrt{\gamma}} \|\varphi\|_{\mathbb{H}_n^{1,\gamma}}, \quad \varphi \in \mathbb{H}_n^{1,\gamma}, \quad (5.47)$$

which follows from the inequality

$$\sup_{z \geq 0} |\varphi_i(z)| \leq \frac{2}{\sqrt{\gamma}} \|\varphi_i\|_{H^{1,\gamma}},$$

proved in [2], see Lemma 4.4.

**Proof of Proposition 4.6:** (A) First we show that (D2) and (D4) imply linear growth of  $\frac{d}{dz}G$ .

$$\begin{aligned} \left\| \frac{d}{dz} G(t, \varphi)(x_i) \right\|_{L^{2,\gamma}}^2 &= \int_0^{+\infty} [g'_z(t, z, x_i, L_t, \varphi(z)) + \langle \nabla g_i(t, z, L_t, \varphi(z)), \varphi'(z) \rangle]^2 e^{\gamma z} dz \\ &\leq 2 \int_0^{+\infty} |h(z)|^2 e^{\gamma z} dz + 2 \sup_{t,z,l,r} \|\nabla g_i(t, z, x_i, l, r)\|^2 \int_0^{+\infty} \|\varphi'(z)\|^2 e^{\gamma z} dz \\ &\leq 2\|h\|_{L^{2,\gamma}}^2 + 2C^2 \|\varphi\|_{\mathbb{H}_n^{1,\gamma}}^2. \end{aligned}$$

To show linear growth of  $\frac{d}{dz}F$  recall that the assumptions in (a) or in (b) imply that

$$J''\left(\int_0^z g_i(t, v, L_t, \varphi(u)) dv\right), \quad J'\left(\int_0^z g_i(t, v, L_t, \varphi(u)) dv\right),$$

are bounded on  $\mathbb{R}$ , so in view of the formula (5.46) we additionally need to show that  $g_i^2$  has linear growth. By (B3) we have

$$\int_0^{+\infty} |g_i^2(t, z, L_t, \varphi(z))|^2 e^{\gamma z} dz \leq C^4 \int_0^{+\infty} \|\varphi(z)\|^2 e^{\gamma z} dz \leq C^4 \|\varphi\|_{\mathbb{H}_n^{1,\gamma}}^2,$$

and linear growth of  $F$  follows. Now we will prove Lipschitz estimates. In view of (D1), (D4) and (5.47) we have

$$\begin{aligned} \left\| \frac{d}{dz} \left( G(t, \varphi)(x_i) - G(t, \phi)(x_i) \right) \right\|_{L^{2,\gamma}}^2 &\leq 2 \int_0^{+\infty} [g'_z(t, z, x_i, L_t, \varphi(z)) - g'_z(t, z, x_i, L_t, \phi(z))]^2 e^{\gamma z} dz \\ &\quad + 2 \int_0^{+\infty} [\langle \nabla g_i(t, z, L_t, \varphi(z)), \varphi'(z) \rangle - \langle \nabla g_i(t, z, L_t, \phi(z)), \phi'(z) \rangle]^2 e^{\gamma z} dz \\ &\leq 2C^2 \|\varphi - \phi\|_{\mathbb{H}_n^{1,\gamma}}^2 + 4 \int_0^{+\infty} \left[ \langle \nabla g_i(t, z, L_t, \varphi(z)), \varphi'(z) - \phi'(z) \rangle \right]^2 e^{\gamma z} dz \\ &\quad + 4 \int_0^{+\infty} \left[ \langle \phi'(z), \nabla g_i(t, z, L_t, \varphi(z)) - \nabla g_i(t, z, L_t, \phi(z)) \rangle \right]^2 e^{\gamma z} dz \end{aligned}$$

$$\begin{aligned}
&\leq 2C^2 \|\varphi - \phi\|_{\mathbb{H}_n^{1,\gamma}}^2 + 4C^2 \|\varphi - \phi\|_{\mathbb{H}_n^{1,\gamma}}^2 \\
&+ 4 \sup_{t,z,l,r,\bar{r}} \frac{|g'_r(t,z,x_i,r) - g'_r(t,z,x_i,\bar{r})|^2}{\|r - \bar{r}\|^2} \int_0^{+\infty} \|\phi'(z)\|^2 \cdot \|\varphi(z) - \phi(z)\|^2 e^{\gamma z} dz \\
&\leq \left( 6C^2 + \frac{16}{\gamma} C^2 \|\phi\|_{\mathbb{H}_n^{1,\gamma}}^2 \right) \|\varphi - \phi\|_{\mathbb{H}_n^{1,\gamma}}^2,
\end{aligned}$$

and local Lipschitz property of  $\frac{d}{dz}G$  follows. It follows from (5.46) that

$$\left\| \frac{d}{dz} \left( F(t, \varphi)(x_i) - F(t, \phi)(x_i) \right) \right\|_{L^{2,\gamma}}^2 \leq 3(I_1(x_i) + I_2(x_i) + I_3(x_i)),$$

where

$$\begin{aligned}
I_1(x_i) &:= \left\| J'' \left( \int_0^z g_i(t, u, L_t, \varphi(u)) du \right) g_i^2(t, z, L_t, \varphi(z)) - J'' \left( \int_0^z g_i(t, u, L_t, \phi(u)) du \right) g_i^2(t, z, L_t, \phi(z)) \right\|_{L^{2,\gamma}}^2 \\
I_2(x_i) &:= \left\| J' \left( \int_0^z g_i(t, u, L_t, \varphi(u)) du \right) g'_z(t, z, x_i, L_t, \varphi(z)) - J' \left( \int_0^z g_i(t, u, L_t, \phi(u)) du \right) g'_z(t, z, x_i, L_t, \phi(z)) \right\|_{L^{2,\gamma}}^2 \\
I_3(x_i) &:= \left\| J' \left( \int_0^z g_i(t, u, L_t, \varphi(u)) du \right) \langle \nabla g_i(t, z, L_t, \varphi(z)), \varphi'(z) \rangle - J' \left( \int_0^z g_i(t, u, L_t, \phi(u)) du \right) \langle \nabla g_i(t, z, L_t, \phi(z)), \phi'(z) \rangle \right\|_{L^{2,\gamma}}^2
\end{aligned}$$

We have

$$\begin{aligned}
I_1(x_i) &\leq 2 \left\| J'' \left( \int_0^z g_i(t, u, L_t, \varphi(u)) du \right) \cdot |g_i^2(t, z, L_t, \varphi(z)) - g_i^2(t, z, L_t, \phi(z))| \right\|_{L^{2,\gamma}}^2 \\
&+ 2 \left\| g_i^2(t, z, L_t, \phi(z)) \cdot \left| J'' \left( \int_0^z g_i(t, u, L_t, \varphi(u)) du \right) - J'' \left( \int_0^z g_i(t, u, L_t, \phi(u)) du \right) \right| \right\|_{L^{2,\gamma}}^2.
\end{aligned}$$

It follows from (LC), (B1) and (B2) that

$$|g_i^2(t, z, L_t, r) - g_i^2(t, z, L_t, \bar{r})| \leq 2C\hat{g} \|r - \bar{r}\|$$

and thus the first expression above can be estimated. For the second we use again (B2) and the fact that the assumptions in (Aa) or (Ab) imply that  $J'''$  is locally bounded on a positive half-line or around zero, respectively. The estimate for  $I_2(x_i)$  follows from (D1) and (D3) while (D1) and (D4) imply local Lipschitz property for  $I_3(x_i)$ .

(B) The proof is similar to part (A). The only difference is that the assumptions on the Lévy measure ensure that  $J', J'', J'''$  are bounded on  $[0, +\infty)$ .  $\square$

## 5.2 Monotonicity of the forward rates

In this section we present the results on positivity, monotonicity and pointwise monotonicity of the solution to (3.16). Our final aim is to prove Proposition 4.3 and Proposition 4.4.

We start with an auxiliary result on positivity and monotonicity of a general system of equations of the form

$$\begin{aligned}
dX_1 &= (AX_1 + F_1(X_1, X_2, \dots, X_n))dt + G_1(X_1, X_2, \dots, X_n)dW, \\
dX_2 &= (AX_2 + F_2(X_1, X_2, \dots, X_n))dt + G_2(X_1, X_2, \dots, X_n)dW, \\
&\vdots \\
dX_n &= (AX_n + F_n(X_1, X_2, \dots, X_n))dt + G_n(X_1, X_2, \dots, X_n)dW,
\end{aligned} \tag{5.48}$$

where  $W$  is a one dimensional Wiener process and

$$F_i, G_i : H^n \rightarrow H, \quad i = 1, 2, \dots, n,$$

with the space  $H$  of square integrable functions on some measurable space  $E$  with a sigma-finite measure. The solution to (5.48) is assumed to be an element of  $H^n$ . The following is a version of the Milian result, see [8].

$C(E)$ ,  $C_c^\infty(E)$  below stand for the space of continuous functions and smooth functions with compact support respectively.

**Theorem 5.2 (Milian)** *Assume that  $A$  generates a strongly continuous semigroup  $S_t, t \geq 0$  in  $H$  and that the semigroup preserves positivity. Assume that for each  $R > 0$  there exists a constant  $C_R$  such that*

$$\|F_i(x) - F_i(y)\|_H + \|G_i(x) - G_i(y)\|_H \leq C_R \|x - y\|_{H^n}, \quad i = 1, 2, \dots, n, \tag{5.49}$$

for each  $x, y \in B_R := \{z \in H^n : \|z\|_{H^n} \leq R\}$ . Assume that there exists a solution  $X$  to (5.48).

a) Let  $X(0) \geq 0$ . If for each  $f \in H_+ \cap C_c^\infty(E)$  and  $\varphi = (\varphi_1, \dots, \varphi_n)$ ,  $\varphi_i \in H_+ \cap C(E)$ ,  $i = 1, 2, \dots, n$  such that  $\langle \varphi_i, f \rangle = 0$  for some  $i = 1, 2, \dots, n$  the following holds

$$\langle F_i(\varphi), f \rangle \geq 0, \tag{5.50}$$

$$\langle G_i(\varphi), f \rangle = 0, \tag{5.51}$$

then  $X(t) \geq 0, t \geq 0$ . Moreover, if  $X(t) \geq 0, t \geq 0$  then (5.50) and (5.51) hold.

b) Let  $X_1(0) \geq X_2(0) \geq \dots \geq X_n(0)$ . If for each  $f \in H_+ \cap C_c^\infty(E)$  and  $\varphi = (\varphi_1, \dots, \varphi_n)$ ,  $\varphi_i \in H \cap C(E)$ ,  $i = 1, 2, \dots, n$  such that  $\varphi_1 \geq \varphi_2 \geq \dots \geq \varphi_n$  and  $\langle \varphi_i, f \rangle = \langle \varphi_{i+1}, f \rangle$  for some  $i = 1, 2, \dots, n-1$  the following holds

$$\langle F_i(\varphi), f \rangle \geq \langle F_{i+1}(\varphi), f \rangle \tag{5.52}$$

$$\langle G_i(\varphi), f \rangle = \langle G_{i+1}(\varphi), f \rangle, \tag{5.53}$$

then  $X_i(t) \geq X_{i+1}(t), t \geq 0, i = 1, 2, \dots, n-1$ . Moreover, if  $X_i(t) \geq X_{i+1}(t), t \geq 0, i = 1, 2, \dots, n-1$  then (5.52) and (5.53) hold.

In the original formulation condition (5.49) is replaced by a global Lipschitz condition. Conditions for positivity of a general SPDE under locally Lipschitz conditions were proven in [2], see Theorem 4.2. Monotonicity under locally Lipschitz conditions can be shown in a similar way.

Using Theorem 5.2 we characterize monotonicity and positivity of solutions to the HJMM equation with Lévy noise.

**Theorem 5.3** Assume that the transformations  $G, F$  given by (3.17) and (3.19) are locally Lipschitz in  $\mathbb{L}_n^{2,\gamma}$  and let  $r(t), t \geq 0$ , be a solution to the HJMM equation in the space  $\mathbb{L}_n^{2,\gamma}$ . The following statements hold.

- a) If  $r_0 \geq 0$ , then  $r(t) \geq 0, t \geq 0$  if and only if both conditions (P1) and (P2) are satisfied.
- b) Assume that  $r_0(z, x_i)$  is decreasing in  $x_i \in I$ . Then  $r(t, z, x_i)$  is decreasing in  $x_i \in I$  if and only if both conditions (M1) and (M2) are satisfied.

**Proof of Theorem 5.3:** The solution to (3.16) is positive and monotone if and only if for each  $\varepsilon \in (0, 1)$  is the solution  $r^\varepsilon(t)$  of the system

$$\begin{aligned} dr^\varepsilon(t, z, x_i) = & \left( Ar^\varepsilon(t, z, x_i) + F(t, r^\varepsilon)(z, x_i) + (a - m_\varepsilon)G(t, r^\varepsilon)(z, x_i) \right) dt \\ & + qG(t, r^\varepsilon)(z, x_i)dW(t) + G(t, r^\varepsilon)(z, x_i)dP^\varepsilon(t), \end{aligned} \quad (5.54)$$

with

$$m_\varepsilon := \int_{\{\varepsilon < y < 1\}} y\nu(dy), \quad P^\varepsilon(t) = \int_0^t \int_{\{y > \varepsilon\}} y\pi(ds, dy).$$

Equation (5.54) arises from the original equation (3.16) by cutting out compensated jumps smaller than  $\varepsilon$ . Since  $P^\varepsilon$  is a compound Poisson process, it has a finite number of jumps on each finite time interval and between the jumps the driving noise is the Wiener process. Thus to get positivity and monotonicity one can directly use Theorem 5.2.

(a) First we show necessity of (P1). Fix any  $x_i \in I$ . Condition (5.51) applied with  $\varphi = (\varphi_1, \dots, \varphi_n)$ , s.t.  $\varphi_i \equiv 0$  provides, that for any  $f \in H_+ \cap C_c^\infty(\mathbb{R}_+)$  we have

$$\int_0^{+\infty} g_i(t, z, L_t, \varphi(z))f(z)e^{\gamma z}dz = 0,$$

which implies  $g_i(t, z, L_t, \varphi(z)) = 0, z \geq 0$ . This gives (P1).

To show sufficiency of (P1) we will check conditions (5.50) and (5.51). Let  $f \in H_+ \cap C_c^\infty(\mathbb{R}_+)$  and  $\varphi = (\varphi_1, \dots, \varphi_n)$ ,  $\varphi \in H_+ \cap C(\mathbb{R}_+)$  be such that

$$\int_0^{+\infty} \varphi_i(z)f(z)e^{\gamma z}dz = 0,$$

for some  $i = 1, 2, \dots, n$ . Then

$$\lambda(A_i \cap B) = 0, \quad \text{where } A_i := \{z : \varphi_i(z) > 0\}, \quad B := \{z : f(z) > 0\},$$

and  $\lambda$  stands for the Lebesgue measure. Using (P1) we have

$$\begin{aligned} & \langle F(t, \varphi)(x_i) + (a - m_\varepsilon)G(t, \varphi)(x_i), f \rangle \\ &= \int_0^{+\infty} J' \left( \left( \int_0^z g_i(t, v, L_t, \varphi(v))dv \right) + a - m_\varepsilon \right) g_i(t, z, L_t, \varphi(z))f(z)e^{\gamma z}dz \\ &= \int_{A_i} J' \left( \left( \int_0^z g_i(t, v, L_t, \varphi(v))dv \right) + a - m_\varepsilon \right) g_i(t, z, L_t, \varphi(z))f(z)e^{\gamma z}dz \\ &+ \int_B J' \left( \left( \int_0^z g_i(t, v, L_t, \varphi(v))dv \right) + a - m_\varepsilon \right) g_i(t, z, L_t, \varphi(z))f(z)e^{\gamma z}dz = 0, \end{aligned}$$

because both integrals are equal to zero. Moreover, it holds

$$\begin{aligned} |\langle G(t, \varphi)(x_i), f \rangle| &\leq \int_0^{+\infty} |g_i(t, z, L_t, \varphi(z))| |f(z)e^{\gamma z} dz \\ &\leq \int_{A_i} |g_i(t, z, L_t, \varphi(z))| |f(z)e^{\gamma z} dz + \int_B |g_i(t, z, L_t, \varphi(z))| |f(z)e^{\gamma z} dz = 0. \end{aligned}$$

At the moment of jump of  $P^\varepsilon$  the solution remains positive if

$$r_i + g_i(t, z, L_t, r)u \geq 0, \quad t, z \geq 0, r \in \mathbb{R}^n, r \geq 0, u \in \text{supp}\{\nu\} \cap (\varepsilon, +\infty), i = 1, 2, \dots, n.$$

Passing to the limit  $\varepsilon \downarrow 0$  we obtain (P2).

(b) To show necessity of (M1) let us examine condition (5.53). Then for  $\varphi = (\varphi_1, \dots, \varphi_n)$  with  $\varphi_i \in H \cap C(\mathbb{R}_+)$ ,  $\forall i$  such that  $\varphi_i = \varphi_{i+1}$  for some  $i$  and any  $f \in H_+ \cap C_c^\infty(\mathbb{R}_+)$  holds

$$\int_0^{+\infty} (g_{i+1}(t, z, L_t, \varphi(z)) - g_i(t, z, L_t, \varphi(z))) f(z) e^{\gamma z} dz = 0,$$

which is equivalent to (M1). To show sufficiency of (M1), consider  $\varphi = (\varphi_1, \dots, \varphi_n)$ ,  $\varphi_i \in H \cap C(\mathbb{R}_+)$ ,  $\forall i$ ,  $f \in H_+ \cap C_c^\infty(\mathbb{R}_+)$  such that  $\varphi_i \geq \varphi_{i+1}$ ,  $\forall i$  and

$$\int_0^{+\infty} (\varphi_i(z) - \varphi_{i+1}(z)) f(z) e^{\gamma z} dz = 0, \quad \text{for some } i = 1, \dots, n.$$

Then

$$\lambda(B \cap \{z : \varphi_i(z) > \varphi_{i+1}(z)\}) = 0, \quad \text{where } B := \{z : f(z) > 0\},$$

and thus we have

$$\begin{aligned} &\langle F(t, \varphi)(x_{i+1}) + (a - m_\varepsilon)G(t, \varphi)(x_{i+1}), f \rangle \\ &= \int_0^{+\infty} J' \left( \left( \int_0^z g_{i+1}(t, v, L_t, \varphi(v)) dv \right) + a - m_\varepsilon \right) g_{i+1}(t, z, L_t, \varphi(z)) f(z) e^{\gamma z} dz \\ &= \int_B J' \left( \left( \int_0^z g_{i+1}(t, v, L_t, \varphi(v)) dv \right) + a - m_\varepsilon \right) g_{i+1}(t, z, L_t, \varphi(z)) f(z) e^{\gamma z} dz \\ &= \int_0^{+\infty} J' \left( \left( \int_0^z g_i(t, v, L_t, \varphi(v)) dv \right) + a - m_\varepsilon \right) g_i(t, z, L_t, \varphi(z)) f(z) e^{\gamma z} dz \\ &= \langle F(t, \varphi)(x_i) + (a - m_\varepsilon)G(t, \varphi)(x_i), f \rangle \end{aligned}$$

and

$$\begin{aligned} \langle G(t, \varphi)(x_{i+1}), f \rangle &= \int_B g_{i+1}(t, z, L_t, \varphi(z)) f(z) e^{\gamma z} dz = \int_B g_i(t, z, L_t, \varphi(z)) f(z) e^{\gamma z} dz \\ &= \langle G(t, \varphi)(x_i), f \rangle. \end{aligned}$$

Hence conditions (5.52) and (5.53) are satisfied. At the moments of jumps the solution remains monotone if for each consecutive pair of coordinates  $i, i+1$  holds

$$\begin{aligned} r_i + g_i(t, z, L_t, r)u &\geq r_{i+1} + g_{i+1}(t, z, L_t, r)u, \\ t, z &\geq 0, r \in \mathbb{R}^n, r_i \geq r_{i+1}, \forall i, u \in \text{supp}\{\nu\} \cap (\varepsilon, +\infty). \end{aligned}$$

Letting  $\varepsilon \downarrow 0$  yields (M2). □

Now we pass to the pointwise monotonicity of the solution to the HJMM equation taking values in  $\mathbb{L}_n^{2,\gamma}$ . Here we adopt the idea presented in [11] and instead of studying irregular functions of  $z$

$$z \rightarrow r(t, z, x_i),$$

which are defined only for almost each  $z$ , we consider the functions

$$z \rightarrow \int_0^t r(s, z, x_i) ds,$$

which we prove to be well defined for each  $z \geq 0$  and regular. The following Proposition 5.4 and Proposition 5.5 lead to the proof of Proposition 4.3.

**Proposition 5.4** *Let  $r(t), t \geq 0$  be a solution of the HJMM equation in  $\mathbb{L}_n^{2,\gamma}$  with coefficients  $F$  and  $G$  which are locally Lipschitz and have linear growth. Assume that  $Z$  is a square integrable Lévy process, that is*

$$\int_{\{|y|>1\}} |y|^2 \nu(dy) < +\infty. \quad (5.55)$$

Then for each  $z \geq 0$ ,  $x_i \in I$  and  $t \geq 0$  the function

$$z \rightarrow \int_0^t r(s, z, x_i) ds,$$

is well defined. Moreover, for each  $z \geq 0$  and a sequence  $z_n \xrightarrow[n]{} z$  there exists a subsequence  $z_{n_k}, k = 1, 2, \dots$  such that

$$\int_0^t r(s, z_{n_k}, x_i) ds \xrightarrow[k]{} \int_0^t r(s, z, x_i) ds. \quad (5.56)$$

**Proof:** For the sake of brevity we use the notation  $F_i(t, r(t))(z) := F(t, r(t))(z, x_i)$ ,  $G_i(t, r(t-))(z) := G(t, r(t-))(z, x_i)$ . Integrating both sides of (3.15) and using the form of the semigroup  $S$  we obtain

$$\begin{aligned} \int_0^t r(s, z, x_i) ds &= \int_0^t r(0, z + s, x_i) ds + \int_0^t \int_0^s F_i(u, r(u))(z + s - u) du ds \\ &\quad + \int_0^t \int_0^s G_i(u, r(u-))(z + s - u) dZ(u) ds. \end{aligned} \quad (5.57)$$

Now we will argue that the Fubini theorem and the stochastic Fubini theorem can be applied. We have

$$\begin{aligned} \int_0^t \int_z^{z+t-u} |F_i(u, r(u))(v)| dv du &\leq \int_0^t \int_0^{+\infty} |F_i(u, r(u))(v)| dv du \\ &\leq \frac{1}{\sqrt{\gamma}} \int_0^t \|F_i(u, r(u))\|_{L^{2,\gamma}} du \leq \frac{C}{\sqrt{\gamma}} \int_0^t (1 + \|r(u)\|_{\mathbb{L}_n^{2,\gamma}}) du < +\infty, \end{aligned}$$

where the last inequality follows from the fact that  $r(t), t \geq 0$  is càdlàg in  $\mathbb{L}_n^{2,\gamma}$  and thus also bounded on bounded intervals. Further, we have

$$\begin{aligned} E \int_0^t \int_z^{z+t-u} |G_i(u, r(u-))(v)|^2 dv du &\leq E \int_0^t \int_0^{+\infty} |G_i(u, r(u-))(v)|^2 dv du \\ &\leq E \int_0^t \|G_i(u, r(u-))\|_{L^{2,\gamma}}^2 du \leq C^2(x_i) \int_0^t E(1 + \|r(u-)\|_{\mathbb{L}_n^{2,\gamma}})^2 du < +\infty, \end{aligned}$$

where the last inequality follows from the fact  $\sup_{u \in [0,t]} E\|r(u)\|_{\mathbb{L}_n^{2,\gamma}}^2 < +\infty$ , see Theorem 9.29 in [10]. Applying the deterministic and stochastic Fubini theorems in (5.57), see Theorem 8.14 in [10], we obtain

$$\begin{aligned} \int_0^t r(s, z, x_i) ds &= \int_z^{z+t} r(0, s, x_i) ds + \int_0^t \int_z^{z+t-u} F_i(u, r(u))(v) dv du \\ &\quad + \int_0^t \int_z^{z+t-u} G_i(u, r(u-))(v) dv dZ(u). \end{aligned} \quad (5.58)$$

Since the right hand side is uniquely defined for each  $z \geq 0$ , the first part of the assertion follows. Now we show (5.56). It is clear that

$$z \longrightarrow \int_z^{z+t} r(0, s, x_i) ds$$

is continuous. Since

$$z \rightarrow \int_z^{z+t-u} F_i(u, r(u))(v) dv$$

is continuous and for each  $z$

$$\left| \int_z^{z+t-u} F_i(u, r(u))(v) dv \right| \leq \int_0^{+\infty} |F_i(u, r(u))(v)| dv$$

with

$$\int_0^t \int_0^{+\infty} |F_i(u, r(u))(v)| dv du < +\infty,$$

it follows from the dominated convergence theorem that the function

$$z \rightarrow \int_0^t \int_z^{z+t-u} F_i(u, r(u))(v) dv du$$

is continuous. Consider any  $z \geq 0$  and a sequence  $z_n \rightarrow z$ . Then for  $\varepsilon > 0$  and sufficiently large  $n$  holds

$$\begin{aligned} E \int_0^t \left| \int_{z_n}^{z_n+t-u} G_i(u, r(u-))(v) dv - \int_z^{z+t-u} G_i(u, r(u-))(v) dv \right|^2 du \\ \leq 4E \int_0^t \left( \int_{z-\varepsilon}^{z+t+\varepsilon} |G_i(u, r(u-))(v)| dv \right)^2 du \\ \leq 4(2\varepsilon + t)E \int_0^t \int_0^{+\infty} |G_i(u, r(u-))(v)|^2 dv du < +\infty. \end{aligned}$$



Since the function

$$z \rightarrow \int_z^{z+t-u} G_i(u, r(u-))(v) dv$$

is continuous it follows that

$$E \int_0^t \left| \int_{z_n}^{z_n+t-u} G_i(u, r(u-))(v) dv - \int_z^{z+t-u} G_i(u, r(u-))(v) dv \right|^2 du \xrightarrow[n]{} 0.$$

That condition and (5.55) imply that

$$E \left( \int_0^t \int_{z_n}^{z_n+t-u} G_i(u, r(u-))(v) dv dZ(u) - \int_0^t \int_z^{z+t-u} G_i(u, r(u-))(v) dv dZ(u) \right)^2 \rightarrow 0,$$

and thus we can find a subsequence  $z_{n_k}$  such that

$$\int_0^t \int_{z_{n_k}}^{z_{n_k}+t-u} G_i(u, r(u-))(v) dv dZ(u) \xrightarrow[k]{} \int_0^t \int_z^{z+t-u} G_i(u, r(u-))(v) dv dZ(u)$$

almost surely. This leads to (5.56).  $\square$

**Proposition 5.5** *Let  $r(t), t \geq 0$  be a càdlàg nonnegative process taking values in  $\mathbb{L}_n^{2,\gamma}$ . Then for each  $x_i \in I$  we have*

$$\int_0^t r(s, z, x_i) ds \geq 0,$$

for each  $t \geq 0$  and almost all  $z \geq 0$ .

**Proof:** Since the process  $r(t), t \geq 0$  is càdlàg, it follows that for each  $x_i \in I$  the function  $t \rightarrow \|r(t, \cdot, x_i)\|_{L^{2,\gamma}}$  is bounded on bounded intervals. Thus for any  $0 \leq a < b \leq +\infty$  we have

$$\begin{aligned} \int_0^t \int_a^b r(s, z, x_i) dz ds &\leq \int_0^t \int_0^{+\infty} r(s, z, x_i) e^{\frac{\gamma}{2}z} e^{-\frac{\gamma}{2}z} dz ds \\ &\leq \int_0^t \left( \int_0^{+\infty} r^2(s, z, x_i) e^{\gamma z} dz \right)^{\frac{1}{2}} \left( \int_0^{+\infty} e^{-\gamma z} dz \right)^{\frac{1}{2}} ds \leq \frac{1}{\sqrt{\gamma}} \int_0^t \|r(s, \cdot, x_i)\|_{L^{2,\gamma}} ds < +\infty. \end{aligned}$$

Thus the Fubini theorem yields

$$0 \leq \int_0^t \int_a^b r(s, z, x_i) dz ds = \int_a^b \int_0^t r(s, z, x_i) ds dz.$$

Since  $a, b$  are arbitrary, the assertion follows.  $\square$

Now we can easily prove Proposition 4.3.

**Proof of Proposition 4.3:** From monotonicity of the solution and Proposition 5.5 follows that for each  $t \geq 0$  and  $x_i \in I$

$$\int_0^t r(s, z, x_i) ds \geq \int_0^t r(s, z, x_{i+1}) ds \quad \text{for almost all } z \geq 0.$$

In view of Proposition 5.4 the inequality above holds for each  $z \geq 0$ . As a consequence, for each  $z \geq 0$  we have

$$r(t, z, x_i) \geq r(t, z, x_{i+1}), \quad \text{for almost all } t \geq 0.$$

□

**Proof of Proposition 4.4:** Since  $r$  is a solution of the HJMM equation, it follows that the process  $\hat{P}(t, T, x_i)$  is a local martingale for each  $T > 0, x_i \in I$ . From the positivity of  $r$  follows

$$0 \leq \hat{P}(t, T, x_i) = e^{-\int_0^t r(s, 0, 1) ds} \mathbf{1}_{\{L_t \leq x_i\}} e^{-\int_0^{T-t} r(t, u, x_i) du} \leq 1,$$

and thus  $\hat{P}(t, T, x_i)$  is a martingale. From the identity

$$\hat{P}(T, T, x_i) = e^{-\int_0^T r(s, 0, 1) ds} \mathbf{1}_{\{L_T \leq x_i\}},$$

follows that

$$\hat{P}(t, T, x_i) = E(e^{-\int_0^T r(s, 0, 1) ds} \mathbf{1}_{\{L_T \leq x_i\}} \mid \mathcal{F}_t),$$

and as a consequence one obtains

$$P(t, T, x_i) = e^{\int_0^t r(s, 0, 1) ds} E(e^{-\int_0^T r(s, 0, 1) ds} \mathbf{1}_{\{L_T \leq x_i\}} \mid \mathcal{F}_t) = E(e^{-\int_t^T r(s, 0, 1) ds} \mathbf{1}_{\{L_T \leq x_i\}} \mid \mathcal{F}_t).$$

Thus monotonicity of  $P(t, T, x_i)$  in  $x_i$  follows. Now assume to the contrary that

$$r(t, 0, x_i) < r(t, 0, x_{i+1}) \quad \text{for some } t \geq 0, L_t \leq x_i, i \in \{1, 2, \dots, n-1\}.$$

Since  $r(t) \in \mathbb{H}_n^{1, \gamma}$  the function  $z \rightarrow r(t, z, x_i)$  is continuous and thus

$$r(t, u, x_i) < r(t, u, x_{i+1}), \quad u \in (0, \varepsilon),$$

for some  $\varepsilon > 0$ . This implies

$$P(t, t + \varepsilon, x_i) = e^{-\int_0^\varepsilon r(t, u, x_i) du} > e^{-\int_0^\varepsilon r(t, u, x_{i+1}) du} = P(t, t + \varepsilon, x_{i+1})$$

which is a contradiction to monotonicity of  $P(t, T, x_i)$  in  $x_i$  proved above. □

### 5.3 Proof of Proposition 4.10 and calculations for the Example 4.11

**Proof of Proposition 4.10:** (A) (a) It follows from the condition (P2) and positivity of  $g_i$  that

$$u \geq -\frac{r_i}{g_i(t, z, l, r)}, \quad t, z, r \geq 0, l \in I, u \in \text{supp}\{\nu\}, i = 1, 2, \dots, n.$$

It follows from (P1) that

$$u \geq -\frac{r_i}{g_i(t, z, l, r) - g_i(t, z, l, \mathbf{1}_i(r))}.$$

Passing with  $r \downarrow 0$  and taking supremum over  $t, z, l, r$  yields (4.28).

(b) It follows from (4.28) that

$$u \geq -\frac{1}{g'_{r_i}(t, z, x_i, l, \mathbf{1}_i(r))}, \quad t, z \geq 0, r \geq 0, l \in I, u \in \text{supp}\{\nu\}, i = 1, 2, \dots, n.$$

Using (4.29) yields

$$u \geq -\frac{1}{g'_{r_i}(t, z, x_i, l, \mathbf{1}_i(r))} \geq -\frac{r_i}{g_i(t, z, l, r)}$$

which is (P2).

(B) (a) In view of (M1) condition (M2) is equivalent to

$$\left( \frac{g_{i+1}(t, z, l, r_1, \dots, r_i, r_{i+1}, \dots, r_n) - g_{i+1}(t, z, l, r_1, \dots, r_{i+1}, r_{i+1}, \dots, r_n)}{r_i - r_{i+1}} - \frac{g_i(t, z, l, r_1, \dots, r_i, r_{i+1}, \dots, r_n) - g_i(t, z, l, r_1, \dots, r_{i+1}, r_{i+1}, \dots, r_n)}{r_i - r_{i+1}} \right) u \leq 1,$$

for  $r \in \mathbb{R}^n, r_1 \geq r_2 \geq \dots \geq r_n, t, z \geq 0, l \in I, u \in \text{supp}\{\nu\}, i = 1, 2, \dots, n-1$ . Passing to the limit  $r_i \downarrow r_{i+1}$  yields (4.30). (4.31) can be obtained in a similar way.

(b) The concavity of  $g_{i+1}(t, z, l, r) - g_i(t, z, l, r)$  in  $r_i$  and (M1) imply

$$\frac{g_{i+1}(t, z, l, r_1, \dots, r_i, r_{i+1}, \dots, r_n) - g_i(t, z, l, r_1, \dots, r_i, r_{i+1}, \dots, r_n)}{r_i - r_{i+1}} \leq \frac{d}{dr_i} [g_{i+1}(t, z, l, r_1, \dots, r_{i+1}, r_{i+1}, \dots, r_n) - g_{i+1}(t, z, l, r_1, \dots, r_{i+1}, r_{i+1}, \dots, r_n)], \quad (5.59)$$

while convexity gives

$$\frac{g_{i+1}(t, z, l, r_1, \dots, r_i, r_{i+1}, \dots, r_n) - g_i(t, z, l, r_1, \dots, r_i, r_{i+1}, \dots, r_n)}{r_i - r_{i+1}} \geq \frac{d}{dr_i} [g_{i+1}(t, z, l, r_1, \dots, r_{i+1}, r_{i+1}, \dots, r_n) - g_{i+1}(t, z, l, r_1, \dots, r_{i+1}, r_{i+1}, \dots, r_n)]. \quad (5.60)$$

If  $\text{supp}\{\nu\} \subseteq (0, +\infty)$ , then multiplying both sides of (5.59) with  $u > 0$  and using (4.30) yields (M2). Thus (i) implies (M2). If  $\text{supp}\{\nu\} \subseteq (-\infty, 0)$ , then multiplying both sides of (5.60) with  $u < 0$  and using (4.30) yields (M2) and shows sufficiency of (ii). If (iii) holds, then (M2) is satisfied for each  $u < 0$ . For  $u > 0$  we can argue as in (1). Similarly (iv) implies (M2) for  $u > 0$  while for  $u < 0$  we use the same argument as in (ii).  $\square$

**Calculations for the Example 4.11** (a) It is clear that (4.33) implies (P1). We have

$$\frac{d}{dr_i} g_i(t, z, l, r) = f_1(t) f_2(z) f_3(l) h_1(r_1) \dots h_{i-1}(r_{i-1}) \left[ h'_i(r_i) h(r_i) + h_i(r_i) h'(r_i) \right] h_{i+1}(r_{i+1}) \dots h_n(r_n),$$

which gives

$$\frac{d}{dr_i} g_i(t, z, l, r) = f_1(t) f_2(z) f_3(l) h_1(r_1) \dots h_{i-1}(r_{i-1}) \left[ h_i(0) h'(0) \right] h_{i+1}(r_{i+1}) \dots h_n(r_n) \geq 0,$$

for  $r_i = 0$ . Thus (4.35) implies necessary condition (4.28). Further (4.33) and (4.34) imply inequality

$$h_i(r_i) h(r_i) \leq h_i(0) h'(0) r_i, \quad r_i \geq 0,$$

which is exactly (4.29). Hence it follows from Proposition 4.10 (A) that (P2) is satisfied.

(b) It is clear that (M1) holds. If  $r_i = r_{i+1}$ , the following holds

$$\frac{d}{dr_i} (g_{i+1}(t, z, l, r) - g_i(t, z, l, r)) = -f_1(t) f_2(z) f_3(l) h_1(r_1) \dots h_n(r_n) h'(r_i),$$

and thus condition (4.30) is implied by (4.36) and (4.37). Moreover, we have

$$g_{i+1}(t, z, l, r) - g_i(t, z, l, r) = f_1(t) f_2(z) f_3(l) h_1(r_1) \dots h_n(r_n) [h(r_{i+1}) - h(r_i)], \quad r_1 \geq r_2 \geq \dots r_n,$$

and thus for convexity in  $r_i$  we need to examine convexity of the function

$$r_i \rightarrow h_i(r_i)[h(r_{i+1}) - h(r_i)], \quad r_i \geq r_{i+1},$$

In view of (4.32), (4.33) and (4.36) we have

$$\frac{d^2}{dr_i^2} \left[ h_i(r_i)[h(r_{i+1}) - h(r_i)] \right] = h_i''(r_i)[h(r_{i+1}) - h(r_i)] - 2h_i'(r_i)h'(r_i) - h_i(r_i)h''(r_i) \geq 0,$$

and convexity follows. It is also clear that  $g_{i+1}(t, z, l, r) \leq g_i(t, z, l, r)$ . Thus, in view of Proposition 4.10 (B), condition (M2) is satisfied.  $\square$

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### References

- [1] Barski, M., Zabczyk, J. : "Forward rate models with linear volatilities", (2012), *Finance and Stochastics* 16, 3, p. 537-560;
- [2] Barski, M., Zabczyk, J. : "Heath-Jarrow-Morton-Musiela equation with Lévy perturbation", (2012), *Journal of Differential Equations*, 253, 9, p. 2657-2697;
- [3] Filipović, D., Overbeck, L., Schmidt, T. : "Dynamic CDO Term Structure Modelling", (2011), *Mathematical Finance* 21, 53-71,
- [4] Filipović, D., Tappe, S. : "Existence of Lévy term structure models", (2008), *Finance and Stochastics*, 12, (1), 83-115.
- [5] Filipović, D., Tappe, S., Teichmann, J. : "Term structure models driven by Wiener processes and Poisson measures: Existence and positivity", (2010), *SIAM Journal on Financial Mathematics* 1, 523-554,
- [6] Heath, D., Jarrow, R., Morton, A.: Bond pricing and the term structure of interest rates: a new methodology for contingent claim valuation, *Econometrica*, (1992), 60, 77-105,
- [7] Marinelli, C.: Local well-posedness of Musiela SPDE's with Lévy noise, (2010), *Mathematical Finance*, 20, 341-363,
- [8] Milian, A. : "Comparison theorems for stochastic evolution equation", (2002), *Stochastics and Stochastics Reports*, 72, 79-108;
- [9] Musiela, M. "Stochastic PDEs and term structure models", (1993), *Journées International de Finance, IGR-AFFI, La Baule*.
- [10] Peszat, Sz., Zabczyk, J.: "Stochastic partial differential equations with Lévy noise", (2007), Cambridge University Press;
- [11] Rusinek, A.: "A note on HJMM models on a space of square integrable functions", (2012), manuscript,
- [12] Rusinek, A.: Invariant measures for forward rate HJM model with Lévy noise, Preprint IMPAN 669 (2006), <http://www.impan.pl/Preprints/p669.pdf>,
- [13] Sato, K.I.: "Lévy Processes and Infinite Divisible Distributions", (1999), Cambridge University Press;
- [14] Schmidt T., Tappe S. : "Dynamic term structure modelling with default and mortality risk: new results on existence and monotonicity", (2015), *Banach Center Publications* 105, 211-238,
- [15] Schmidt T., Zabczyk J. : "CDO term structure modeling with Lévy processes and the relation to market models", (2012), *International Journal of Theoretical and Applied Finance*, 15, No. 1.